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ELASTIC CONTACT BETWEEN A SPHERE AND A SEMI INFINITE TRANSVERSELY ISOTROPIC BODY

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Abstract—The problem is solved by using a Hankel transformation. The stress and displacement expressions are explicitly given for any point of the medium. Curves of numerical results are presented.

1. NOTATION

The following symbols are used in this paper:

r, θ, z cylindrical co-ordinates
σ_{ij} stress components
ϵ_{ij} strain components
$\gamma_{ij} = 2\epsilon_{ij}$
u_r, u_θ, w displacements
J_ν Bessel functions
$\mathcal{H}_0$ Hankel transformation
P contact force
R radius of sphere
E, σ elastic constants of sphere
a radius of the area of contact
p_0 applied stress in centre of contact
a_{ij} elastic coefficients of anisotropic half-space

$$a' = \frac{a_{13}(a_{11} - a_{12})}{a_{11}a_{33} - a_{13}^2}$$

$$b = \frac{a_{13}(a_{13} + a_{44}) - a_{12}a_{33}}{a_{11}a_{33} - a_{13}^2}$$

$$c = \frac{a_{13}(a_{11} - a_{12}) + a_{11}a_{44}}{a_{11}a_{33} - a_{13}^2}$$

$$d = \frac{a_{11}^2 - a_{12}^2}{a_{11}a_{33} - a_{13}^2}$$

$$s_1 = \left(\frac{a' + c + \sqrt{[(a' + c)^2 - 4d]}}{2d}\right)^{1/2}$$

$$s_2 = \left(\frac{a' + c - \sqrt{[(a' + c)^2 - 4d]}}{2d}\right)^{1/2}$$

$$\rho_1 = 1 - a's_1^2$$

$$\rho_2 = 1 - a's_2^2$$

$$q_1 = (b - a's_2^2)\rho_1$$

$$q_2 = (b - a's_1^2)\rho_2$$

$$\nu = \frac{(b - 1)\sqrt{d}}{a'c - d}$$

$$\mu = \frac{(b - 1)(a' + \sqrt{d})}{a'c - d}$$

$$\delta_1 = a_{44}\frac{s_1 s_2}{s_2 - s_1} + (a_{12} - a_{11})\frac{\nu s_1^2 \rho_2}{s_2 - s_1}$$

$$\delta_2 = a_{44}\frac{s_1 s_2}{s_2 - s_1} + (a_{12} - a_{11})\frac{\nu s_2^2 \rho_1}{s_2 - s_1}.$$

$$k_i = \frac{s_i^2}{(s_2 - s_1)\sqrt{d}}$$

$$l_1 = \frac{\nu s_1^2 \rho_2}{s_2 - s_1}$$

$$l_2 = \frac{\nu s_2^2 \rho_1}{s_2 - s_1}$$

2. INTRODUCTION

The contact problem in isotropic elastic media is now well understood and solutions to a large number of cases have been obtained. In the case of anisotropic media the basic equations are much more complicated and few results are known. In this paper we consider the so-called Hertz problem, of finding the stresses and displacements produced inside two elastic bodies when they are pressed together. More precisely we consider the elastic punching between a transversely isotropic half-space and an isotropic sphere. The punching is executed without friction and in the direction of the axis of elastic symmetry of the medium, so that we have an axially symmetric problem.

The problem of a concentrated normal load on a transversely isotropic half-space was solved by Lekhnitski[2].

Willis[4] considered this Hertz problem for anisotropic bodies. Although he did not find a complete analytic solution to this problem, he determined the area of contact and the pressure distribution between the bodies.

Here we present a simple and complete analytic solution to the transversely isotropic contact problem.

In Section 3, the basic equations are stated together with the boundary conditions. The solution is given in Section 4.

In the last section, we show the influence of this anisotropy.

3. BASIC EQUATIONS

Let us consider a homogeneous transversely isotropic half-space whose surface lies in the horizontal (r, θ) plane and whose axis of elastic symmetry (z) is vertical. For this material, there are five elastic coefficients.

The stress-strain relationship associated with frictionless axi-symmetric loading are:

$$\left.\begin{aligned}\epsilon_{rr} &= a_{11}\sigma_{rr} + a_{12}\sigma_{\theta\theta} + a_{13}\sigma_{zz},\\ \epsilon_{\theta\theta} &= a_{12}\sigma_{rr} + a_{11}\sigma_{\theta\theta} + a_{13}\sigma_{zz},\\ \epsilon_{zz} &= a_{13}\sigma_{rr} + a_{13}\sigma_{\theta\theta} + a_{33}\sigma_{zz},\\ \gamma_{rz} &= a_{44}\sigma_{rz},\end{aligned}\right\} \tag{1}$$

where the strain components are defined as:

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \epsilon_{\theta\theta} = \frac{u_r}{r}, \qquad \epsilon_{zz} = \frac{\partial w}{\partial z}, \qquad \gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial w}{\partial r}. \tag{2}$$

The equations of equilibrium are:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \tag{3}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0, \tag{4}$$

and the equations of compatibility transformed by relations (1) are:

$$(a_{12} - a_{11})(\sigma_{\theta\theta} - \sigma_{rr}) - r\frac{\partial}{\partial r}(a_{11}\sigma_{\theta\theta} + a_{12}\sigma_{rr} + a_{13}\sigma_{zz}) = 0, \tag{5}$$

$$\frac{\partial}{\partial z^2}(a_{12}\sigma_{\theta\theta} + a_{11}\sigma_{rr} + a_{13}\sigma_{zz}) + \frac{\partial^2}{\partial r^2}(a_{13}\sigma_{\theta\theta} + a_{13}\sigma_{rr} + a_{33}\sigma_{zz}) - a_{44}\frac{\partial^2 \sigma_{rz}}{\partial r \partial z} = 0. \tag{6}$$

The boundary conditions are taken as:

(i) r or z infinite

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{rz} = 0. \tag{7}$$

(ii) On the horizontal surface $(z = 0)$

$$\sigma_{rz} = 0, \qquad \sigma_{zz} = \begin{cases} -p(r) & r \leq a \\ 0 & r \geq a \end{cases} \tag{8}$$

where $p(r)$ is the pressure distribution on the area of contact. This pressure has been determined already[1–3]

$$p(r)=\frac{p_0}{a}\sqrt{(a^2-r^2)} \tag{9}$$

with:

$$a=\left[\frac{3PR}{4}\left(\frac{\delta_1-\delta_2}{2}+\frac{1-\sigma^2}{E}\right)\right]^{1/2} \quad \text{and} \quad p_0=\frac{3P}{2\pi a^2}. \tag{10}$$

δ_1, δ_2 are defined in Section 1.

4. GENERAL SOLUTION

(a) *Determination of an intermediate function* φ

As it is described for example in [5]. We can show easily, by substitution, that the eqns (3), (5) and (6) are identically satisfied if we introduce an intermediate function φ such that:

$$\left.\begin{aligned}
\sigma_{rr}&=-\frac{\partial}{\partial z}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{b}{r}\frac{\partial\varphi}{\partial r}+a'\frac{\partial^2\varphi}{\partial z^2}\right),\\
\sigma_{\theta\theta}&=-\frac{\partial}{\partial z}\left(b\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+a'\frac{\partial^2\varphi}{\partial z^2}\right),\\
\sigma_{zz}&=\frac{\partial}{\partial z}\left(c\frac{\partial^2\varphi}{\partial r^2}+\frac{c}{r}\frac{\partial\varphi}{\partial r}+d\frac{\partial^2\varphi}{\partial z^2}\right),\\
\sigma_{rz}&=\frac{\partial}{\partial r}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+a'\frac{\partial^2\varphi}{\partial z^2}\right).
\end{aligned}\right\} \tag{11}$$

In this case, the corresponding values of the displacements are:

$$\left.\begin{aligned}
u_r&=-(1-b)(a_{11}-a_{12})\frac{\partial^2\varphi}{\partial r\partial z},\\
u_\theta&=0,\\
w&=a_{44}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)+(a_{33}d-2a_{13}a')\frac{\partial^2\varphi}{\partial z^2}.
\end{aligned}\right\} \tag{12}$$

From the second equation of equilibrium (4), we obtain the following partial differential equation:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+a'\frac{\partial^2\varphi}{\partial z^2}\right)+\frac{\partial^2}{\partial z^2}\left(c\frac{\partial^2\varphi}{\partial r^2}+\frac{c}{r}\frac{\partial\varphi}{\partial r}+d\frac{\partial^2\varphi}{\partial z^2}\right)=0. \tag{13}$$

Using the boundary conditions, we can now completely solve the problem. We look for φ in the form of a product:

$$\varphi=Z(tz)\cdot J_0(tr), \tag{14}$$

where t is a dimensional parameter. We shall eliminate t by an integration. By substituting (14) into (13), we obtain for Z the linear differential equation:

$$dZ^{(4)}(tz)-(a'+c)Z''(tz)+Z(tz)=0. \tag{15}$$

The stresses which are determined by the function φ must satisfy the conditions at infinite (7); then we have:

$$\varphi=(C(t)e^{-s_1tz}+D(t)\,e^{-s_2tz})J_0(tr). \tag{16}$$

The function φ obtained by integrating expression (16) with respect to t from 0 to ∞ is also a solution of eqn (13). From now on we use the following expression which will also be called φ:

$$\varphi(r, z) = \int_0^\infty [C(t)\,e^{-s_1 tz} + D(t)\,e^{-s_2 tz}] \cdot J_0(tr)\,dt. \tag{17}$$

The stresses σ_{zz} and σ_{rz}, found by means of formulae (11) are also represented in the form of integrals:

$$\left.\begin{aligned}\sigma_{zz} &= -\int_0^\infty [C(t)\,e^{-s_1 tz} s_1(ds_1^2 - c) + D(t)\,e^{-s_2 tz} s_2(ds_2^2 - c)]t^3 J_0(tr)\,dt,\\ \sigma_{rz} &= -\int_0^\infty [C(t)\,e^{-s_1 tz}(a's_1^2 - 1) + D(t)\,e^{-s_2 tz}(a's_2^2 - 1)]t^3 J_1(tr)\,dt.\end{aligned}\right\} \tag{18}$$

We have seen (9) the pressure distribution on the area of contact is equal to:

$$\rho(r) = \begin{cases}(\rho_0/a)\sqrt{(a^2 - r^2)} & r \leqslant a\\ 0 & r \geqslant a\end{cases}. \tag{19}$$

If $g(t)$ is the function obtained by the Hankel transformation of $\rho(r)$, we have:

$$g(t) = \mathscr{H}_0(\rho(r)) = \rho_0\left(\frac{\sin at}{at^3} - \frac{\cos at}{t^2}\right) \tag{20}$$

and also:

$$\rho(r) = \mathscr{H}_0(g(t)) = \rho_0 \int_0^\infty \left(\frac{\sin at}{at^3} - \frac{\cos at}{t^2}\right) tJ_0(tr)\,dt. \tag{21}$$

By satisfying boundary conditions (8), relations (18) for $z = 0$, which are identical to the expression (21) of $\rho(r)$, give the following system:

$$\left.\begin{aligned}&Cs_1(ds_1^2 - c) + Ds_2(ds_2^2 - c) = \frac{\rho_0}{t^2}\left(\frac{\sin at}{at^3} - \frac{\cos at}{t^2}\right),\\ &C(a's_1^2 - 1) + D(a's_2^2 - 1) = 0.\end{aligned}\right\} \tag{22}$$

Hence:

$$\left.\begin{aligned}C &= -\frac{\rho_0}{t^2}\left(\frac{\sin at}{at^3} - \frac{\cos at}{t^2}\right)\frac{\rho_2\sqrt{d}}{(s_1 - s_2)(a'c - d)},\\ D &= \frac{\rho_0}{t^2}\left(\frac{\sin at}{at^3} - \frac{\cos at}{t^2}\right)\frac{\rho_1\sqrt{d}}{(s_1 - s_2)(a'c - d)}.\end{aligned}\right\} \tag{23}$$

Then, the final formula for the intermediate function is:

$$\varphi(r, z) = \rho_0 \frac{\sqrt{d}}{(s_1 - s_2)(a'c - d)} \int_0^\infty (-\rho_2\,e^{-s_1 tz} + \rho_1\,e^{-s_2 tz})\left(\frac{\sin at}{at^3} - \frac{\cos at}{t^2}\right)\frac{J_0(tr)}{t^2}\,dt. \tag{24}$$

(b) *Stress and displacement distributions in the medium*

Let us substitute expression (24) into formulae (11) and (12). We obtain stress and displacement expressions in the form of integrals:

$$
\left.\begin{aligned}
\sigma_{rr} &= \rho_0\left\{-\frac{1}{(s_1-s_2)\sqrt{d}}\int_0^\infty (s_1\,e^{-s_1tz}-s_2\,e^{-s_2tz})\left(\frac{\sin at}{at^2}-\frac{\cos at}{t}\right)J_0(tr)\,dt\right.\\
&\left.\quad+\frac{\nu}{r(s_1-s_2)}\int_0^\infty (s_1\rho_2\,e^{-s_1tz}-s_2\rho_1\,e^{-s_2tz})\left(\frac{\sin at}{at^3}-\frac{\cos at}{t^2}\right)J_1(tr)\,dt\right\},\\
\sigma_{\theta\theta} &= \rho_0\left\{\frac{\sqrt{d}}{(s_1-s_2)(a'c-d)}\int_0^\infty (s_1q_2\,e^{-s_1tz}-s_2q_1\,e^{-s_2tz})\left(\frac{\sin at}{at^2}-\frac{\cos at}{t}\right)J_0(tr)\,dt\right.\\
&\left.\quad-\frac{\nu}{r(s_1-s_2)}\int_0^\infty (s_1\rho_2\,e^{-s_1tz}-s_2\rho_1\,e^{-s_2tz})\left(\frac{\sin at}{at^3}-\frac{\cos at}{t^2}\right)J_1(tr)\,dt\right\}\\
\sigma_{zz} &= \frac{\rho_0}{s_1-s_2}\int_0^\infty (s_2\,e^{-s_1tz}-s_1\,e^{-s_2tz})\left(\frac{\sin at}{at^2}-\frac{\cos at}{t}\right)J_0(tr)\,dt,\\
\sigma_{rz} &= \frac{\rho_0}{(s_1-s_2)\sqrt{d}}\int_0^\infty (e^{-s_1tz}-e^{-s_2tz})\left(\frac{\sin at}{at^2}-\frac{\cos at}{t}\right)J_1(tr)\,dt,\\
u_r &= \frac{\rho_0\nu(a_{12}-a_{11})}{s_1-s_2}\int_0^\infty (s_1\rho_2\,e^{-s_2tz}-s_2\rho_1\,e^{-s_2tz})\left(\frac{\sin at}{at^3}-\frac{\cos at}{t^2}\right)J_1(tr)\,dt,\\
w &= \rho_0\int_0^\infty (\delta_1\,e^{-s_1tz}-\delta_2\,e^{-s_2tz})\left(\frac{\sin at}{at^3}-\frac{\cos at}{t^2}\right)J_0(tr)\,dt.
\end{aligned}\right\}\quad(25)
$$

Let us introduce the following notations:

$$
\left.\begin{aligned}
C_{i,j} &= \int_0^\infty \left(\frac{\sin at}{at^j}-\frac{\cos at}{at^{j-1}}\right)J_0(tr)\,e^{-s_itz}\,dt\\
&\qquad\qquad i=1,2 \qquad \text{and} \qquad j=2,3\\
D_{i,j} &= \int_0^\infty \left(\frac{\sin at}{at^j}-\frac{\cos at}{t^{j-1}}\right)J_1(tr)\,e^{-s_itz}\,dt.
\end{aligned}\right\}\quad(26)
$$

Then, the formulae for stresses and displacements take the form:

$$
\left.\begin{aligned}
\sigma_{rr} &= \rho_0\left\{\frac{-1}{(s_1-s_2)\sqrt{d}}(s_1C_{1,2}-s_2C_{2,2})+\frac{\nu}{r(s_1-s_2)}(s_1\rho_2D_{1,3}-s_2\rho_1D_{2,3})\right\},\\
\sigma_{\theta\theta} &= \rho_0\left\{\frac{\sqrt{d}}{(s_1-s_2)(a'c-d)}(s_1q_2C_{1,2}-s_2q_1C_{2,2})-\frac{\nu}{r(s_1-s_2)}(s_1\rho_2D_{1,3}-s_2\rho_1D_{2,3})\right\},\\
\sigma_{zz} &= \rho_0\frac{s_2C_{1,2}-s_1C_{2,2}}{s_1-s_2},\\
\sigma_{rz} &= \rho_0\frac{D_{1,2}-D_{2,2}}{(s_1-s_2)\sqrt{d}},\\
u_r &= \rho_0\frac{\nu(a_{12}-a_{11})}{s_1-s_2}(s_1\rho_2D_{1,3}-s_2\rho_1D_{2,3}),\\
w &= \rho_0(\delta_1C_{1,3}-\delta_2C_{2,3}).
\end{aligned}\right\}\quad(27)
$$

With:

$$
\left.\begin{aligned}
S_{i,1} &= \int_0^\infty \frac{\sin at}{t}J_0(tr)\,e^{-s_itz}\,dt,\\
S^1_{i,1} &= \int_0^\infty \frac{\sin at}{t}J_1(tr)\,e^{-s_itz}\,dt,\\
T^1_{i,1} &= \int_0^\infty \frac{\cos at}{t}J_1(tr)\,e^{-s_itz}\,dt,
\end{aligned}\right\}\quad(28)
$$

the integrals $C_{i,j}$ and $D_{i,j}$ are reduced to the following expressions:

$$\left.\begin{aligned}
C_{i,2} &= 1 - \frac{1}{a}(rS^1_{i,1} + s_i z S_{i,1}),\\
C_{i,3} &= \frac{1}{2a}\left(a^2 - \frac{r^2}{2} + s_i^2 z^2\right)S_{i,1} + \frac{3s_i zr}{4a}S^1_{i,1} + \frac{r}{4}T^1_{i,1} - \frac{s_i z}{2},\\
D_{i,2} &= \frac{1}{2a}(rS_{i,1} - s_i z S^1_{i,1} - aT^1_{i,1}),\\
D_{i,3} &= \frac{1}{3a}\left(a^2 - r^2 + \frac{s_i^2 z^2}{2}\right)S^1_{i,1} + \frac{s_i z}{6}T^1_{i,1} - \frac{rs_i z}{2a}S_{i,1} + \frac{r}{3}.
\end{aligned}\right\} \tag{29}$$

We find for integrals (28) the analytic expressions:

$$\left.\begin{aligned}
S_{i,1} &= \operatorname{arc} tg \frac{a - \beta_i}{\alpha_i + s_i z},\\
S^1_{i,1} &= \frac{\beta_i + a}{r},\\
T^1_{i,1} &= \frac{\alpha_i - s_i z}{r},
\end{aligned}\right\} \tag{30}$$

When

$$\left.\begin{aligned}
\gamma_i^4 &= \frac{1}{a^4}(r^2 - a^2 + s_i^2 z^2)^2 + 4\frac{s_i^2 z^2}{a^2},\\
\alpha_i^2 &= \frac{1}{2}(r^2 + s_i^2 z^2 - a^2 + a^2\gamma_i^2),\\
\beta_i &= -\frac{s_i z a}{\alpha_i}.
\end{aligned}\right\} \tag{31}$$

Thus for any point of the medium, we can calculate α_i, β_i, γ_i. Next we determine successively integrals $S_{i,1}$, $S^1_{i,1}$ and $T^1_{i,1}$ and the integrals $C_{i,j}$, $D_{i,j}$. Finally, by means of formulae (27), we find stress tensor and the displacement at every point.

(c) *Distributions along special lines*

(i) *Axis of punching.* With $r = 0$, the stress and displacement expressions (27) reduce to the following relations:

$$\left.\begin{aligned}
\sigma_{rr} = \sigma_{\theta\theta} &= \rho_0\left\{\frac{\mu}{2} - \frac{1}{\sqrt{d}} - \frac{z}{a}\left[\left(k_1 - \frac{l_1}{2}\right)\operatorname{artcg}\frac{a}{s_1 z} - \left(k_2 - \frac{l_2}{2}\right)\operatorname{arctg}\frac{a}{s_2 z}\right]\right\},\\
\sigma_{zz} &= -\rho_0\left\{1 + \frac{s_1 s_2}{s_1 - s_2}\cdot\frac{z}{a}\left\{\operatorname{arctg}\frac{a}{s_1 z} - \operatorname{arctg}\frac{a}{s_2 z}\right)\right\},\\
\sigma_{rz} &= 0,\\
u_r &= 0,\\
w &= \rho_0\left\{\delta_1\left[\left(a + \frac{s_1^2 z^2}{a}\right)\operatorname{arctg}\frac{a}{s_1 z} - s_1 z\right] - \delta_2\left[\left(a + \frac{s_2^2 z^2}{a}\right)\operatorname{arctg}\frac{a}{s_2 z} - s_2 z\right]\right\}.
\end{aligned}\right\} \tag{32}$$

(ii) *Horizontal surface.* For $z = 0$, the integrals $S_{i,1}$ $S^1_{i,1}$ and $T^1_{i,1}$ lead us to consider two possibilities:

$1 - 0 < r \leq a$: points inside the contact area:

We obtain

$$S_{i,1} = \frac{\pi}{2},$$

$$S^1_{i,1} = \frac{a - \sqrt{(a^2 - r^2)}}{r},$$

$$T^1_{i,1} = 0,$$

and thus, we deduce from expressions (27):

$$\left.\begin{aligned}
\sigma_{rr} &= p_0\left[-\frac{1}{\sqrt{d}}\sqrt{\left(1-\frac{r^2}{a^2}\right)} + \frac{\mu a^2}{3r^2}\left(1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right)\right],\\
\sigma_{\theta\theta} &= p_0\left[\left(\mu - \frac{1}{\sqrt{d}}\right)\sqrt{\left(1-\frac{r^2}{a^2}\right)} - \frac{\mu a^2}{3r^2}\left(1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right)\right],\\
\sigma_{zz} &= -p_0\sqrt{\left(1-\frac{r^2}{a^2}\right)},\\
\sigma_{rz} &= 0,\\
u_r &= p_0\mu(a_{12}-a_{11})\frac{a^2}{3r}\left[1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right],\\
w &= p_0\frac{\pi}{4}(\delta_1-\delta_2)\left(a-\frac{r^2}{2a}\right).
\end{aligned}\right\} \quad (33)$$

$2 - r \geq a$, points outside the contact area. In this case, we have:

$$S_{i,1} = \arcsin\frac{a}{r},$$

$$S^1_{i,1} = \frac{a}{r},$$

$$T^1_{i,1} = \frac{\sqrt{(r^2 - a^2)}}{r},$$

Thus:

$$\left.\begin{aligned}
\sigma_{rr} &= -\sigma_{\theta\theta} = p_0\frac{\mu}{3}\cdot\frac{a^2}{r^2} = \frac{\mu P}{3\pi r^2},\\
\sigma_{zz} &= \sigma_{rz} = 0,\\
u_r &= \frac{\mu P}{2\pi r}(a_{12}-a_{11}),\\
w &= \frac{p_0}{2}(\delta_1-\delta_2)\left[\frac{1}{a}\arcsin\frac{a}{2}\cdot\left(a^2-\frac{r^2}{2}\right)+\frac{1}{2}\sqrt{(a^2-r^2)}.\right.
\end{aligned}\right\} \quad (34)$$

We notice that, from the expressions for the stresses, on the surface of the transversely isotropic half-space, the stresses σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} are principal, and outside the area contact we have a pure shear stress state.

Moreover, outside the contact area the stresses are directly proportional to the contact force P and they are independent of the radius R of the sphere.

We have thus the same properties as for the punching of the isotropic half-space (classical Hertz formulae).

5. OBSERVATION

This calculus was indeed performed because we wanted to extend the application field of our two new non-destructive methods to measure whether the elastic limit or the residual stress tensor in any point *inside* an isotropic structure.

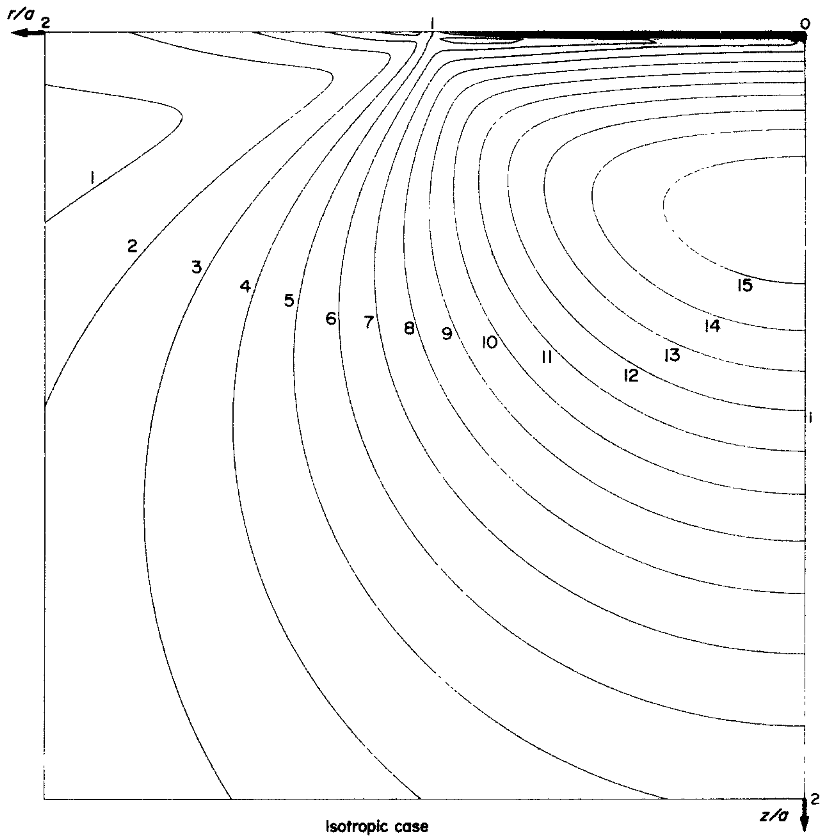

Fig. 1. Isotropic case: $a_{11} = a_{33} = (1/E)$; $a_{12} = a_{13} = -(\sigma/E)$; $a_{44} = 2(a_{11} - a_{12}) = [2(1+\sigma)/E]$; $s_1 = 1$; $s_2 = 1$; $|(\mu/2) - (1/\sqrt{d})| = |(1+2\sigma)/2| < 1$.

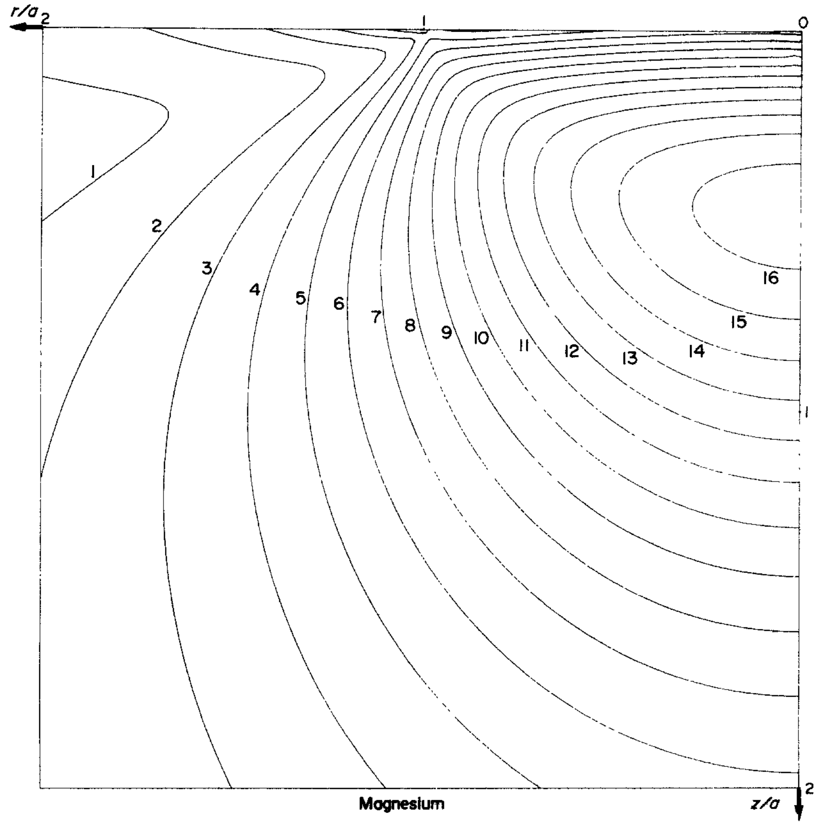

Fig. 2. Magnesium: $a_{11} = 2.21 \times 10^{-11}$ m^2/N; $a_{12} = -7.7 \times 10^{-12}$ m^2/N; $a_{13} = -4.9 \times 10^{-12}$ m^2/N; $a_{33} = 1.97 \times 10^{-11}$ m^2/N; $a_{44} = 6.03 \times 10^{-11}$ m^2/N; $s_1 = 1.388$; $s_2 = 0.705$; $(\mu/2) - (1/\sqrt{d}) = -0.77$.

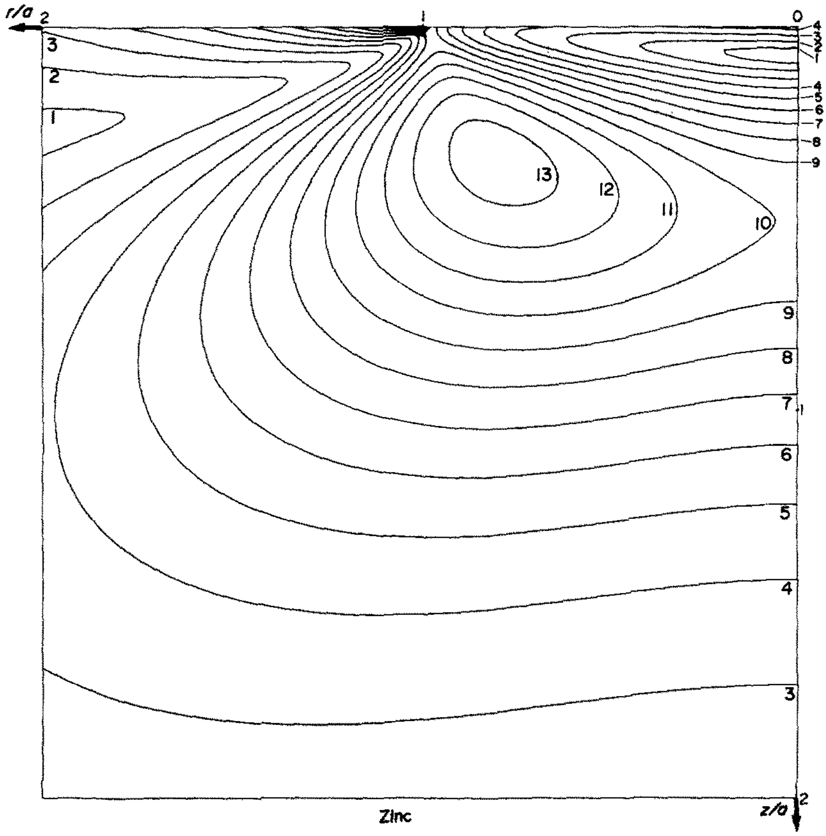

Fig. 3. Zinc: $a_{11} = 8.23 \times 10^{-12}\ m^2/N$; $a_{12} = -3.4 \times 10^{-13}\ m^2/N$; $a_{13} = -6.6 \times 10^{-12}\ m^2/N$; $a_{33} = 2.64 \times 10^{-11}\ m^2/N$; $a_{44} = 2.5 \times 10^{-11}\ m^2/N$; $s_1 = 1.085 + 0.652i$; $s_2 = 1.085 - 0.652i$; $(\mu/2) - (1/\sqrt{d}) = -1.16$.

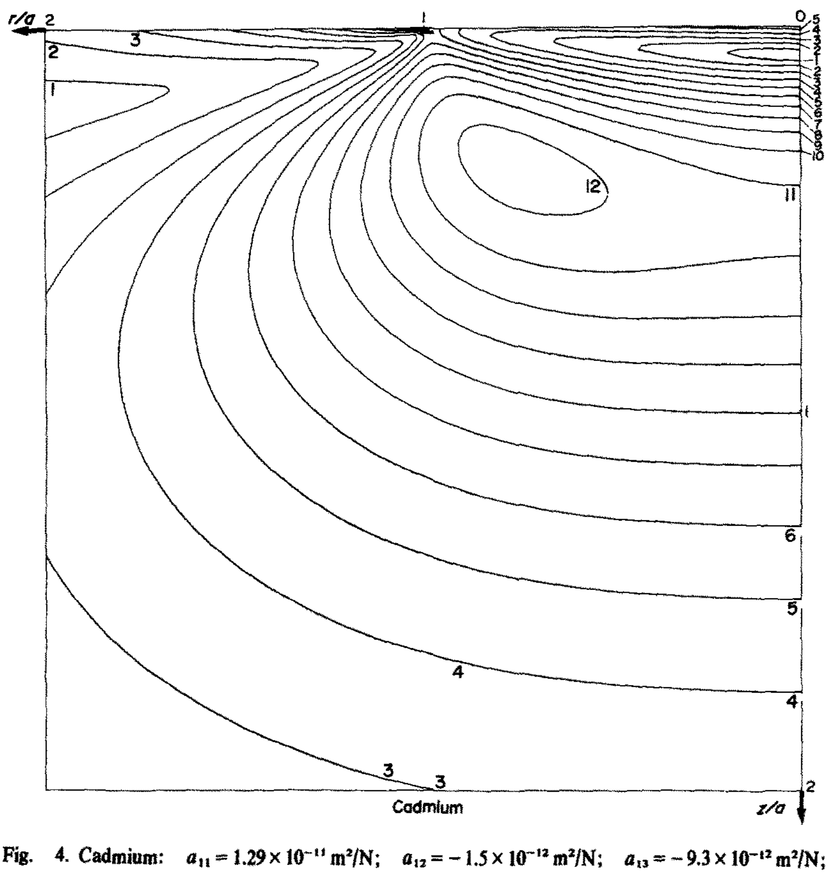

Fig. 4. Cadmium: $a_{11} = 1.29 \times 10^{-11}\ m^2/N$; $a_{12} = -1.5 \times 10^{-12}\ m^2/N$; $a_{13} = -9.3 \times 10^{-12}\ m^2/N$; $a_{33} =$

Hexagonal single crystals are transversely elastically isotropic. Some composite materials and some woods present this kind of anisotropy too. But until now, their plastic properties are not yet well known.

Before doing any theoretical or experimental study for these materials, it was necessary to have the induced stress field for the elastic punching explicitly (see, i.e. [6]).

For the isotropic materials and chiefly for metals, two criteria of plasticity are currently used: Mises's criterion and Tresca's criterion. For anisotropic materials it is not generally possible to use them.

However, just for showing the influence of the anisotropy in the elastic punching, we give here-after the plotting of Mises's criterion for an isotropic material and 3 transversely isotropic materials.

If the inequality:

$$\left|\frac{\mu}{2}-\frac{1}{\sqrt{d}}\right|<1 \tag{35}$$

is satisfied the maximum of the criterion is on the axis of punching. It is off the axis if (35) is not satisfied.

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